Feasible invertibility conditions and MLE of observation-driven models

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Outline

Motivation

- **Why is invertibility needed?**
  1) Ensure the consistency of the MLE.
  2) Uncover the true path of the time varying parameter (even if θ_0 is known).

- **Problem: existing conditions for invertibility are often useless in practice**. This because we need to impose severe restrictions that are unreasonable in empirical applications.

- **Solution:** we derive the consistency of the MLE considering feasible invertibility conditions that can cover situations of practical interest.

Motivation: the model

- Consider the **Beta-t-GARCH model** with leverage effects of Creal et al. (2013) and Harvey (2013)

$$
\begin{aligned}
y_t &= \sqrt{f_t}\varepsilon_t, \quad \varepsilon_t \sim t_v(0,1), \\
f_{t+1} &= \omega + \beta f_t + (\alpha + \gamma d_t)\frac{(v+1)y_t^2}{(v-2) + f_t^{-1} y_t^2},
\end{aligned}
$$

  where $d_t = 1$ if $y_t \leq 0$ and $d_t = 0$ otherwise.

- To ensure the consistency of the MLE, the parameter region Θ where the likelihood is maximized has to satisfy

$$
E \log \left| \beta + (\alpha + \gamma d_t)\frac{(v+1)y_t^4}{\left((v-2)\bar{\omega} + y_t^2\right)^2} \right| < 0, \quad \forall\, \theta \in \Theta.
$$

Motivation: the parameter region

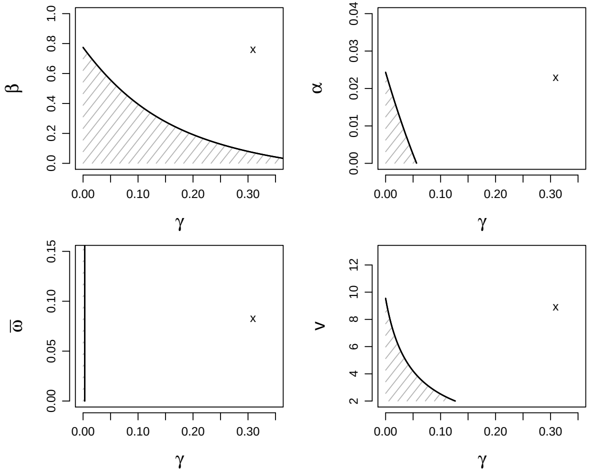

Figure: Invertibility parameter region. The crosses denote the parameter estimate using monthly log-differences of the S&P 500 stock index.

Observation-driven models

- We observe data $\{y_t\}_{t=1}^n$, and we consider the following model

$$y_t|f_t \sim p(y_t|f_t, \theta),$$
$$f_{t+1} = \phi(f_t, y_t, \theta),\ t \in \mathbb{Z},$$

  where $p(\cdot|f_t; \theta)$ is a density function, $\theta \in \Theta$ a parameter vector and ϕ is a continuous function.

- Under correct specification, the data generating process (DGP) satisfies the model equations at $\theta = \theta_0$ and f_t^o denotes the true time varying parameter.

- We are interested in ML estimation of the static parameter θ and, in particular, the consistency of the MLE.

The likelihood function

- Using the observed data, the **filtered parameter** is obtained as

$$\hat{f}_{t+1}(\theta) = \phi(\hat{f}_t(\theta), y_t, \theta), \quad t \in \mathbb{N},$$

for an **initial value** $\hat{f}_1(\theta) \in \mathcal{F}_\theta \subseteq \mathbb{R}$.

- The MLE is then obtained maximizing the likelihood

$$\hat{L}_n(\theta) = n^{-1} \sum_{t=1}^{n} \log p(y_t | \hat{f}_t(\theta), \theta),$$

over the parameter set Θ.

Invertibility

- The filtered parameter $\{\hat{f}_t(\theta)\}_{t\in\mathbb{N}}$ at θ is invertible if

$$\left|\hat{f}_t(\theta) - \tilde{f}_t(\theta)\right| \xrightarrow{a.s.} 0, \qquad \text{as} \;\; t \to \infty.$$

  for any $\hat{f}_1(\theta) \in \mathcal{F}_\theta$, where $\{\tilde{f}_t(\theta)\}_{t\in\mathbb{Z}}$ is a stochastic sequence.

- Invertibility guarantees that the true time varying parameter f_t^o can be recovered, i.e. $|\hat{f}_t(\theta_0) - f_t^o| \xrightarrow{a.s.} 0$.

Invertibility is not merely a technical condition, see Sorokin (2011) and Wintenberger (2013).

How can we ensure invertibility?

- As in Straumann and Mikosh (2006), **sufficient conditions** for invertibility can be obtained on the basis of Theorem 3.1 of Bougerol (1993).

- Bougerol's theorem provides general conditions for stability of stochastic processes.

- We obtain that $\{\hat{f}_t(\theta)\}_{t\in\mathbb{N}}$ is invertible if

$$\boldsymbol{E \log \Lambda_t(\theta) < 0},$$

where

$$\Lambda_t(\theta) = \sup_f \left| \frac{\partial \phi(f, y_t, \theta)}{\partial f} \right| .$$

Invertibility in practice

- In practice $E \log \Lambda_t(\theta) < 0$ **cannot be checked** because $\Lambda_t(\theta)$ depends on the unknown DGP.
- This leads to either a very small region Θ where the likelihood should maximized

$$\hat{\theta}_n = \arg\max_{\theta \in \Theta} \hat{L}_n(\theta),$$

- In practical applications, invertibility is ignored and therefore the **consistency of the MLE is not guaranteed**.

MLE on an empirical region

- To handle this issue, we define the MLE on a parameter region that satisfies an **empirical** version of the **invertibility condition** $E \log \Lambda_t(\theta) < 0$, namely

$$\tilde{\theta}_n = \arg\max_{\theta \in \hat{\Theta}_n} \hat{L}_n(\theta),$$

where

$$\hat{\Theta}_n = \left\{ \theta \in \bar{\Theta} : \frac{1}{n} \sum_{t=1}^{n} \log \Lambda_t(\theta) < 0 \right\}.$$

- Wintenberger (2013) first proposed the estimation of the parameter region for the QMLE of the EGARCH(1,1) model.

Consistency of the MLE

We consider the following conditions:

(C.1) The data generating process is stationary with $E \log \Lambda_t(\theta_0) < 0$.

(C.2) The model is identifiable.

(C.3) The $\log \Lambda_t(\theta)$ is a.s. continuous and it has a finite first moment.

(C.4) The log-likelihood function is uniformly continuous with respect to $\hat{f}_t(\theta)$.

(C.5) The first moment of the likelihood function is uniformly bounded.

Theorem

Let conditions (C.1)-(C.5) hold, then the MLE $\tilde{\theta}_n$ is strongly consistent, i.e.

$$\tilde{\theta}_n \xrightarrow{a.s.} \theta_0, \qquad n \to \infty.$$

Furthermore, $|\hat{f}_n(\tilde{\theta}_n) - f_n^o| \xrightarrow{a.s.} 0$ as n goes to infinity.

Example 1: the model

- The **Beta-t-GARCH model** with leverage effects of Creal et al. (2013) and Harvey (2013) is

$$
\begin{aligned}
y_t &= \sqrt{f_t}\varepsilon_t, \quad \varepsilon_t \sim t_v(0,1), \\
f_{t+1} &= \omega + \beta f_t + (\alpha + \gamma d_t)\frac{(v+1)y_t^2}{(v-2) + f_t^{-1} y_t^2},
\end{aligned}
$$

  where $d_t = 1$ if $y_t \leq 0$ and $d_t = 0$ otherwise.

- The invertibility condition $E \log \Lambda_t(\theta) < 0$ is given by

$$
E \log \left| \beta + (\alpha + \gamma d_t)\frac{(v+1)y_t^4}{\left((v-2)\bar{\omega} + y_t^2\right)^2} \right| < 0, \quad \forall\, \theta \in \Theta.
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Example 1: the parameter region

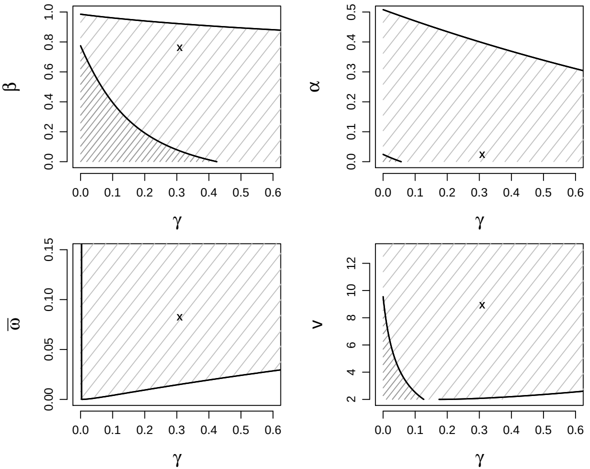

Figure: Invertibility regions obtained considering the monthly log-differences of the S&P 500 stock index.

Example 2: the model

- The **dynamic autoregressive model** of Blasques et al. (2014) and Delle Monache and Petrella (2016) is

$$y_t = f_t y_{t-1} + \sigma \varepsilon_t, \quad \varepsilon_t \sim t_v,$$
$$f_{t+1} = \omega + \beta f_t + \alpha \frac{(y_t - f_t y_{t-1}) y_{t-1}}{1 + v^{-1}\sigma^{-2}(y_t - f_t y_{t-1})^2},$$

- The invertibility condition $E \log \Lambda_t(\theta) < 0$ is given by

$$E \log \max \left\{ \left| \beta - \alpha y_t^2 \right|, \left| \beta + \frac{\alpha}{8} y_t^2 \right| \right\} < 0, \quad \forall\, \theta \in \Theta.$$

- Sufficient conditions leads to a degenerate region with $\alpha = 0$.

Example 2: the parameter region

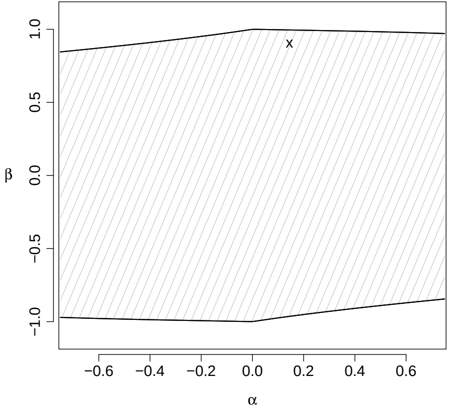

Figure: Invertibility region obtained considering the monthly log-differences of the US unemployment claims.

Example 3: the model

- The **fat-tailed location model** of Harvey and Luati (2014) is given by

$$y_t = f_t + \sigma\varepsilon_t, \quad \varepsilon_t \sim t_v,$$
$$f_{t+1} = \omega + \beta f_t + \alpha \frac{y_t - f_t}{1 + v^{-1}\sigma^{-2}(y_t - f_t)^2},$$

- The coefficient $\Lambda_t(\theta)$ is available in closed form but the expression is quite complicated and therefore not reported.

Example 3: the parameter region

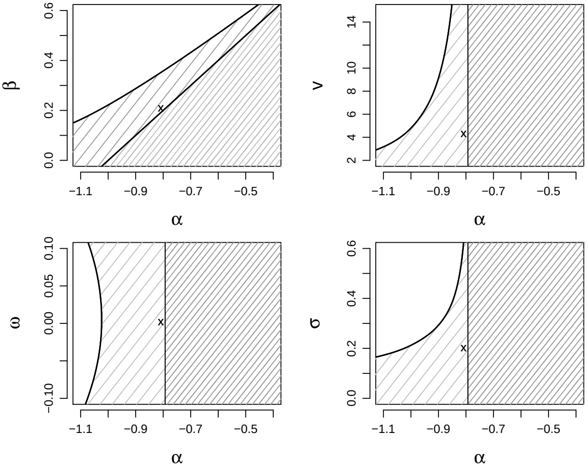

Figure: Invertibility regions obtained considering the monthly differences of the US CP inflation series.

Concluding remarks

- We have derived consistency conditions for the MLE of a wide class of observation-driven time series models.
- The appealing features of our results is that invertibility is feasible to be checked and the theory remains valid also under misspecification.
- The practical relevance of the theory is shown through several practical examples.

Main references

Bougerol, P. (1993). Kalman filtering with random coefficients and contractions. *Control and Optimization* **31**, 942–959.

Straumann, D. & Mikosch, T. (2006). Quasi maximum likelihood estimation in conditionally heteroschedastic time series: a stochastic recurrence equation approach. *Annals of Statistics* **34**, 2449–2495.

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